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1987 J. Phys. A: Math. Gen. 20 4703

(http://iopscience.iop.org/0305-4470/20/14/015)

J. Phys. A: Math. Gen. **20** (1987) 4703-4716. Printed in the UK

On the Nicolai map and Witten index for two-dimensional supersymmetric magnetic field systems

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Received 20 January 1987

Abstract. Various aspects of two-dimensional supersymmetric quantum mechanical systems with magnetic field are studied. In particular, Nicolai maps are obtained and an associated stochastic interpretation in terms of a Markovian process with non-potential forces is given. The evaluation of the Witten index in both the heat kernel and resolvent regularisation scheme is considered. An explicit calculation of a specific model is presented, clarifying how the non-integer value of the index is built up.

1. Introduction

The investigation of supersymmetric quantum mechanical systems is of current interest. They serve as a framework to test and understand specific properties of realistic field theories. The questions considered in this paper are related to, firstly, the stochastic structure in certain supersymmetric theories and, secondly, the breaking of supersymmetry studied via the Witten index.

Close connections between supersymmetry and stochastic processes were first noticed by Parisi and Sourlas [1]. They discovered the supersymmetry of the Lagrangian of a Markovian process with potential forces. On the other hand, Nicolai [2] proposed a characterisation of supersymmetry through the following property of the functional integral: after integrating out the fermionic fields this integral becomes Gaussian by a transformation of the bosonic variables (Nicolai map). The connection between these results was pointed out in [3, 4].

An important example where the Nicolai map is known explicitly for some cases is supersymmetric quantum mechanics. In particular the dependence of the Nicolai map on the boundary conditions (fermion sector structure) has been studied recently [5-7]. In this paper we construct new Nicolai maps for two-dimensional sypersymmetric quantum mechanical systems with magnetic field and show that the associated stochastic process is a Markov process with non-potential forces. To our knowledge the connection between supersymmetry and this type of stochastic process is new.

To study sypersymmetry breaking, Witten [8] introduced a quantity Δ, counting the difference in the number of bosonic and fermionic zero-energy modes. In the presence of a continuous spectrum extending down to zero, this quantity, called the Witten index, has to be regularised. Many authors have studied this (regularised)

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Witten index in supersymmetric quantum mechanics (for a review see [9]). In these studies it was realised that this index can be fractional, for example, due to the occurrence of zero-energy resonances. Recently a new method has been presented to study supersymmetric quantum mechanics, in particular the Witten index, using Krein's spectral shift function [10]. In this paper we present some new remarks about the properties of the Witten index for magnetic field systems in both the heat kernel and resolvent regularisation schemes. We also describe a complete explicit scattering theory calculation of a specific model explaining in detail how the non-integer value of the Witten index is built up.

The rest of this paper is divided into three sections. In § 2 the class of systems considered here is defined and the stochastic process with non-potential forces associated with the Euclidean Schrödinger equation is obtained. In § 3 the functional integral representations of transition amplitudes for Euclidean time are written down. The fermionic degrees of freedom are integrated out and Nicolai maps are found. In addition, for rotationally symmetric magnetic field systems a stochastic process and the Nicolai map associated with the radial Schrödinger equation are given. In § 4 we first consider the existence of a Nicolai map for the Witten index. Next, we discuss the calculation of this index, starting from its path-integral representation, using scaling properties and topological invariance. We finally present a complete scattering theory treatment of a specific rotationally symmetric model. In the appendix we show how the two-dimensional magnetic field system can be obtained from a superspace approach.

2. Stochastic process associated with the Schrödinger equation

In this work we study various properties of the supersymmetric quantum mechanical system [11] with Hamiltonian H and supercharge Q defined by

$$Q=\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \tag{1}$$

$$H=\tfrac{1}{2}Q^2=\frac{1}{2}\begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}\equiv\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{2}$$

with

$$A=-\mathrm{i}\partial_1-a_1(\boldsymbol{q})+\mathrm{i}(\mathrm{i}\partial_2+a_2(\boldsymbol{q})) \tag{3}$$

where $\partial_j\equiv\partial/\partial q^j$, $j=1,2$. Then

$$H_j=\tfrac{1}{2}[(-\mathrm{i}\nabla-\boldsymbol{a})^2-(-1)^j b] \qquad j=1,2 \tag{4}$$

with

$$b(\boldsymbol{q})=(\partial_1 a_2-\partial_2 a_1)(\boldsymbol{q}). \tag{5}$$

Here $\boldsymbol{a}=(a_1,a_2)$ is the vector potential and b is the magnetic field.

In this section a Markovian stochastic process will be associated with H_1 and H_2. It turns out that these processes belong to a class for which, as far as we know, no relation with supersymmetry has been found before.

In [12] Graham has given a method to associate a stochastic process with the Schrödinger equation for a particle in a scalar potential $V(\boldsymbol{q})$ and a magnetic field described by a vector potential $\boldsymbol{a}(\boldsymbol{q})$. Here we use this method and show that this process can be characterised completely, in contrast with the general case, since by supersymmetry explicit knowledge of the ground state is available.

First we consider the general Schrödinger equation (here $\boldsymbol{q} \equiv (q^1, \ldots, q^d)$, $d \in \mathbb{N}$)

$$\mathrm{i}\dot{\psi}(\boldsymbol{q}, t) = [\tfrac{1}{2}(-\mathrm{i}\nabla - \boldsymbol{a}(\boldsymbol{q}))^2 + V(\boldsymbol{q})]\psi(\boldsymbol{q}, t). \tag{6}$$

Let $\psi^{(0)}(\boldsymbol{q})$ be a ground state and let the gauge be uniquely fixed by the requirement that $\psi^{(0)}(\boldsymbol{q})$ is real and positive. In this gauge the ground-state energy is zero, $\psi^{(0)}(\boldsymbol{q})$ can be written as

$$\psi^{(0)}(\boldsymbol{q}) = \exp(-\phi(\boldsymbol{q})) \tag{7}$$

which defines $\phi(\boldsymbol{q})$, and (6) leads to

$$\nabla \cdot \boldsymbol{a} - 2\boldsymbol{a} \cdot \nabla\phi = 0 \tag{8}$$

$$V + \tfrac{1}{2}\boldsymbol{a}^2 = \tfrac{1}{2}[(\nabla\phi)^2 - \nabla^2\phi]. \tag{9}$$

The quantity

$$\Psi(\boldsymbol{q}, t) = \psi^{(0)}(\boldsymbol{q})\psi(\boldsymbol{q}, t) \tag{10}$$

satisfies

$$-\mathrm{i}\dot{\Psi}(\boldsymbol{q}, t) = \tfrac{1}{2}\nabla^2\Psi(\boldsymbol{q}, t) - \nabla \cdot (\boldsymbol{K}(\boldsymbol{q})\Psi(\boldsymbol{q}, t)) \tag{11}$$

where

$$\boldsymbol{K}(\boldsymbol{q}) = -\nabla\phi(\boldsymbol{q}) + \mathrm{i}\boldsymbol{a}(\boldsymbol{q}). \tag{12}$$

With the complex equations (11) and (12) can be associated the following real equations:

$$\dot{W}(\boldsymbol{q}, \tau) = \tfrac{1}{2}\nabla^2 W(\boldsymbol{q}, \tau) - \nabla \cdot (\boldsymbol{K}(\boldsymbol{q}, \lambda) W(\boldsymbol{q}, \tau)) \tag{13}$$

$$\boldsymbol{K}(\boldsymbol{q}, \lambda) = -\nabla\phi(\boldsymbol{q}) - \lambda\boldsymbol{a}(\boldsymbol{q}). \tag{14}$$

If (13) and (14) can be solved for arbitrary λ then $\Psi(\boldsymbol{q}, t)$ can be obtained from $W(\boldsymbol{q}, \tau)$ by the analytic continuation

$$\tau \to \mathrm{i}t \qquad \lambda \to -\mathrm{i}. \tag{15}$$

Equation (13) can be interpreted as the Fokker-Planck equation of a Markovian stochastic process with cartesian diffusion matrix and drift vector $\boldsymbol{K}(\boldsymbol{q}, \lambda)$, $W(\boldsymbol{q}, \tau)$ being the probability density of the random variable $\boldsymbol{q}$ at time τ. For more details and a discussion of the properties and the physical meaning of this stochastic process we refer to [12].

Here we recall that an important and useful classification of Fokker-Planck models is based upon the properties of the stationary drift velocity which in our case, for natural boundary conditions, is given by $-\lambda\boldsymbol{a}(\boldsymbol{q})$. The following three classes of models can be distinguished (see [13] for a systematic treatment):

$$\begin{array}{lll} \text{(i)} & \boldsymbol{a} = 0 & \\ \text{(ii)} & \boldsymbol{a} \neq 0 & \nabla \cdot \boldsymbol{a} = 0 \\ \text{(iii)} & \nabla \cdot \boldsymbol{a} \neq 0. & \end{array} \tag{16}$$

The first class are stochastic processes with potential forces. The first two classes are characterised uniquely by the requirement that the weak noise asymptotic solution coincides with the exact solution. In [1] the connection between models of class (i) and supersymmetry is established. In [14] it is shown that with all supersymmetric quantum mechanical systems defined by Lagrangians constructed from real unconstrained supervariables a stochastic process belonging to this class is associated. Therefore a connection between supersymmetry and stochastic processes with non-potential forces can only be obtained for Lagrangians constructed from constrained supervariables. The systems under consideration here have this property (see the appendix). We indeed find that they are associated with processes of the second class.

A difficulty of the procedure (6)-(14) is that the drift vector of the stochastic process (14) is only completely characterised when the function $\phi(\boldsymbol{q})$ and therefore the ground state $\psi^{(0)}(\boldsymbol{q})$ is known explicitly [12]. It is a nice feature of supersymmetric systems, in particular of $H_j, j=1,2$ ((1)-(5)), that information on possible ground states is available. For these Hamiltonians the Schrödinger equations are

$$\mathrm{i}\dot{\psi}_j(\boldsymbol{q}, t) = \tfrac{1}{2}[(-\mathrm{i}\nabla - \boldsymbol{a}(\boldsymbol{q}))^2 - (-1)^j b(\boldsymbol{q})]\psi_j(\boldsymbol{q}, t) \qquad j=1,2. \tag{17}$$

We write the vector potential as

$$\boldsymbol{a}(\boldsymbol{q}) = (\partial_2\phi(\boldsymbol{q}) + \partial_1\chi(\boldsymbol{q}), -\partial_1\phi(\boldsymbol{q}) + \partial_2\chi(\boldsymbol{q})) \tag{18}$$

with $\phi(\boldsymbol{q})$ and $\chi(\boldsymbol{q})$ arbitrary smooth functions. Since $\chi(\boldsymbol{q})$ is a gauge potential, the magnetic field only depends on $\phi(\boldsymbol{q})$, namely

$$b(\boldsymbol{q}) = -\Delta\phi(\boldsymbol{q}). \tag{19}$$

A candidate for a zero-energy ground-state wavefunction for $H_j, j=1,2$, can be written down immediately in closed form. In the gauge where $\chi \equiv 0$ it is real and given by

$$\psi_j^{(0)}(\boldsymbol{q}) = \exp[(-1)^j\phi(\boldsymbol{q})] \qquad j=1,2. \tag{20}$$

Only if the normalisability condition

$$\int \mathrm{d}\boldsymbol{q}|\psi_j^{(0)}(\boldsymbol{q})|^2 < \infty \tag{21}$$

is satisfied, then $\psi_j^{(0)}$ is an element of the Hilbert space of the quantum system and it represents a true ground state. In the absence of normalisability the associated stochastic process does not have a stationary probability distribution, i.e. it is only defined for a finite time.

From the foregoing results we conclude that the stochastic process associated with the pair of supersymmetric Hamiltonians $H_j, j=1,2$, given by (4), (18) and (19) is a two-dimensional Markovian stochastic process with unit diffusion matrix and drift vector

$$\begin{aligned} K_j^1(\boldsymbol{q}, \lambda) &= (-1)^j\partial_1\phi(\boldsymbol{q}) - \lambda\partial_2\phi(\boldsymbol{q}) \\ K_j^2(\boldsymbol{q}, \lambda) &= (-1)^j\partial_2\phi(\boldsymbol{q}) + \lambda\partial_1\phi(\boldsymbol{q}) \qquad j=1,2. \end{aligned} \tag{22}$$

In the case that $\psi_j^{(0)}(\boldsymbol{q})$ is normalisable, $(-\lambda\partial_2\phi, \lambda\partial_1\phi)$ is the stationary drift velocity. Since the latter is non-zero and its divergence is zero the stochastic process falls into class (ii) (16).

Finally, we remark that an alternative characterisation of the stochastic process with Fokker-Planck equation (13) is given by the stochastic differential equations

$$\dot{\boldsymbol{q}}(\tau) = \boldsymbol{K}(\boldsymbol{q}, \lambda) + \boldsymbol{\eta}(\tau) \tag{23}$$

where $\boldsymbol{\eta}$ is Gaussian white noise (see, e.g., [12]). In the next section it is shown that (23) with $\boldsymbol{K}(\boldsymbol{q}, \lambda)$ given by (22) for $\lambda = -\mathrm{i}$ plays the role of a Nicolai map in the Euclidean functional integral.

3. Nicolai map

In this section we obtain Nicolai maps [2-4] for the supersymmetric system with Hamiltonian H given by (2)-(5).

In order to derive a representation of transition amplitudes by a path integral over commuting and anticommuting coordinates we introduce canonical operators $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{p}}$ and fermionic creation and annihilation operators $\hat{\psi}^+$ and $\hat{\psi}$ satisfying

$$\{\hat{\psi}^+, \hat{\psi}\} = 1. \tag{24}$$

We write H as

$$H = \tfrac{1}{2}(\hat{p}_1 - a_1(\hat{\boldsymbol{q}}))^2 + \tfrac{1}{2}(\hat{p}_2 - a_2(\hat{\boldsymbol{q}}))^2 + \tfrac{1}{2}[\hat{\psi}, \hat{\psi}^+]b(\hat{\boldsymbol{q}}). \tag{25}$$

Then H_1 is the Hamiltonian in the 'bosonic sector' (states annihilated by $\hat{\psi}$) and H_2 is the Hamiltonian in the 'fermionic sector' (states annihilated by $\hat{\psi}^+$).

Following standard methods [15, 16], transition amplitudes of the quantum theory for Euclidean time $\tau = \mathrm{i}t$, i.e.

$$I_K = \langle \boldsymbol{Q}; k|\exp(-\tau H)|\boldsymbol{Q}_0; k\rangle \qquad k = 1, 2 \tag{26}$$

where k denotes the sector, can be represented by the path integral

$$I_k = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \mathrm{d}\boldsymbol{q}_i\right)\left(\frac{1}{2\pi\varepsilon}\right)^{N+1} J_k^N$$
$$\times \exp\left\{-\sum_{j=1}^{N+1} \varepsilon\left[\frac{1}{2}\left(\frac{q_j^\mu - q_{j-1}^\mu}{\varepsilon}\right)^2 - \frac{\mathrm{i}}{2}\left(\frac{q_j^\mu - q_{j-1}^\mu}{\varepsilon}\right)(a^\mu(\boldsymbol{q}_j) + a^\mu(\boldsymbol{q}_{j-1}))\right]\right\} \tag{27}$$

where

$$\varepsilon = \tau/(N+1)$$
$$\boldsymbol{q}_{N+1} = \boldsymbol{Q} \qquad \boldsymbol{q}_0 = \boldsymbol{Q}_0 \tag{28}$$

and

$$J_k^N = \int \left(\prod_{i=0}^{N+1} \mathrm{d}\zeta_i\, \mathrm{d}\zeta_i^*\right) g_k \exp\left\{-\sum_{j=1}^{N+1} \varepsilon\left[\zeta_j^*\left(\frac{\zeta_j - \zeta_{j-1}}{\varepsilon}\right) + \tfrac{1}{2}b(\boldsymbol{q}_j)\zeta_j^*\zeta_{j-1} + \tfrac{1}{2}b(\boldsymbol{q}_{j-1})\zeta_j^*\zeta_j\right]\right\}. \tag{29}$$

Here ζ_i and ζ_i^*, $i = 0, \ldots, N+1$, are pairs of complex conjugate Grassmann variables. The quantities g_k, $k = 1, 2$, containing the boundary conditions are given by [5]

$$g_1 = \exp(-\zeta_0^*\zeta_0) \tag{30}$$

$$g_2 = \exp(-\zeta_0^*\zeta_{N+1}). \tag{31}$$

A shorthand notation for (27)-(31) is

$$I_k = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}} \mathscr{D}\boldsymbol{q}(t')\mathscr{D}\zeta(t')\mathscr{D}\zeta^*(t') \exp\left(-\int_0^\tau L\, \mathrm{d}t'\right) g_k(\zeta(0), \zeta(\tau)) \tag{32}$$

where L is the following supersymmetric Lagrangian:

$$L = L_1 + L_2 \tag{33}$$

$$L_1 = \tfrac{1}{2}\dot{q}^\mu\dot{q}^\mu - \mathrm{i}\dot{q}^\mu a^\mu(\boldsymbol{q}) \tag{34}$$

$$L_2 = \zeta^*(-\partial/\partial t' + b(\boldsymbol{q}))\zeta. \tag{35}$$

First we recall, proceeding formally, the definition of a Nicolai map [2-4]. After integrating out the Grassmann variables $\zeta(t')$ and $\zeta^*(t')$ in (32) one obtains an effective functional integral over commuting variables $\boldsymbol{q}(t')$ only, namely

$$I_k = \int_{Q_0}^{Q} \mathscr{D}\boldsymbol{q}(t') J_k[\boldsymbol{q}(t')] \exp\left(-\int_0^\tau L_1\,\mathrm{d}t'\right) \tag{36}$$

where

$$J_k[\boldsymbol{q}(t')] = \int \mathscr{D}\zeta(t')\mathscr{D}\zeta^*(t') \exp\left(-\int_0^\tau L_2\,\mathrm{d}t'\right) g_k(\zeta(0), \zeta(\tau)). \tag{37}$$

A Nicolai map for the functional integral (32) is then a transformation of variables $\boldsymbol{q} \to \boldsymbol{\eta}$ which fulfils two requirements: (i) L_1 is reduced to a quadratic form (apart from a total time derivative); (ii) the Jacobian of the transformation cancels $J_k[\boldsymbol{q}]$. As a result of the transformation the path integral is brought into Gaussian form apart from non-trivial boundary conditions.

Next, we consider the first requirement. Using (18) the bosonic Lagrangian L_1 can be written as a sum of squares modulo a total derivative in several ways. Writing

$$L_1 = \tfrac{1}{2}[\dot{q}^1 + (-1)^m(\partial_1\phi) - \mathrm{i}(\partial_2\phi)]^2 + \tfrac{1}{2}[\dot{q}^2 + (-1)^m(\partial_2\phi) + \mathrm{i}(\partial_1\phi)]^2 + (-1)^m\dot{\phi} - \mathrm{i}\dot{\chi} \qquad m = 1, 2 \tag{38}$$

one can see that candidates for a Nicolai map are

$$\begin{aligned} \eta^1 &= \dot{q}^1 + (-1)^m(\partial_1\phi) - \mathrm{i}(\partial_2\phi) \\ \eta^2 &= \dot{q}^2 + (-1)^m(\partial_2\phi) + \mathrm{i}(\partial_1\phi) \qquad m = 1, 2. \end{aligned} \tag{39}$$

To prove that (39) is indeed a Nicolai map for I_k with $k = m$ we now proceed to a precise treatment of the second requirement in the discrete representation. By performing the integration over Grassmann variables in (29) one obtains

$$J_k^N = \exp\left(-\frac{1}{2}\varepsilon \sum_{j=1}^{N+1} (-1)^k b(\boldsymbol{q}_j)\right) \tag{40}$$

where we have used the fact that the difference between $\varepsilon b(\boldsymbol{q}_j)$ and $\varepsilon b(\boldsymbol{q}_{j-1})$ is negligible in the limit $N \to \infty$. A possible discrete representation of the maps (39) is

$$\begin{aligned} \eta_j^1 &= (q_j^1 - q_{j-1}^1)/\varepsilon + (-1)^m(\partial_1\phi)(\tilde{\boldsymbol{q}}_j) - \mathrm{i}(\partial_2\phi)(\tilde{\boldsymbol{q}}_j) \\ \eta_j^2 &= (q_j^2 - q_{j-1}^2)/\varepsilon + (-1)^m(\partial_2\phi)(\tilde{\boldsymbol{q}}_j) + \mathrm{i}(\partial_1\phi)(\tilde{\boldsymbol{q}}_j) \qquad j = 1, \ldots, N+1;\ m = 1, 2 \end{aligned} \tag{41}$$

with

$$\tilde{\boldsymbol{q}}_j = \tfrac{1}{2}(\boldsymbol{q}_j + \boldsymbol{q}_{j-1}). \tag{42}$$

The Jacobian of the transformation $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{N+1} \to \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_{N+1}$ through (41) and (42) is

$$\left\|\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\eta}}\right\| = \prod_{j=1}^{N+1} \left|\det \frac{\partial \boldsymbol{\eta}_j}{\partial \boldsymbol{q}_j}\right|^{-1} \tag{43}$$

with

$$\begin{aligned} \det\frac{\partial \boldsymbol{\eta}_j}{\partial \boldsymbol{q}_j} &= \det\begin{vmatrix} 1/\varepsilon + \frac{1}{2}(-1)^m\partial_1\partial_1\phi - \frac{1}{2}\mathrm{i}\partial_1\partial_2\phi & \frac{1}{2}(-1)^m\partial_2\partial_1\phi - \frac{1}{2}\mathrm{i}\partial_2\partial_2\phi \\ \frac{1}{2}(-1)^m\partial_1\partial_2\phi + \frac{1}{2}\mathrm{i}\partial_1\partial_1\phi & 1/\varepsilon + \frac{1}{2}(-1)^m\partial_2\partial_2\phi + \frac{1}{2}\mathrm{i}\partial_2\partial_1\phi \end{vmatrix} \\ &= (1/\varepsilon^2)\exp[-\tfrac{1}{2}\varepsilon(-1)^m b(\tilde{\boldsymbol{q}}_j) + \mathrm{O}(\varepsilon^2)]. \end{aligned} \tag{44}$$

Comparing (40), (43) and (44) one concludes that (41) with $m=1,2$ is a Nicolai map for I_k, $k=1,2$, respectively. (The prefactor $1/\varepsilon^2$ in (44) is absorbed in the volume element and, as before, the difference between $b(\tilde{\boldsymbol{q}}_j)$ in (44) and $b(\boldsymbol{q}_j)$ in (40) does not contribute in the limit $N\to\infty$.)

We remark that in [17] a magnetic field system in four dimensions has been discussed, one of the conclusions being that a Nicolai map only exists in the case of a constant field.

The existence of a Nicolai map is independent of the coordinates used provided that the path integral is defined in a covariant manner (see [16], ch 6). For example, using the methods of [18] it can be shown that the path integral in polar coordinates ($q^1=r\cos\theta$, $q^2=r\sin\theta$)

$$I_k=\int_{r_0,\theta_0}^{r,\theta}\mathscr{D}r\mathscr{D}\theta\mathscr{D}\zeta\mathscr{D}\zeta^*\exp\left(-\int_0^\tau L\,\mathrm{d}t'\right)g_k(\zeta(0),\zeta(\tau)) \tag{45}$$

where

$$L=\tfrac{1}{2}(\dot r^2+r^2\dot\theta^2)-\mathrm{i}(\dot r/r)(\partial_\theta\phi)+\mathrm{i}\dot\theta r(\partial_r\phi)-\mathrm{i}\dot\chi+\zeta^*(-\partial/\partial t'-\Delta\phi)\zeta \tag{46}$$

has the Nicolai map

$$\begin{aligned}&\eta^1\cos\theta+\eta^2\sin\theta=\dot r+(-1)^k(\partial_r\phi)-\mathrm{i}(\partial_\theta\phi)\\&-(1/r)[\eta^1\sin\theta-\eta^2\cos\theta]=\dot\theta+(-1)^k(1/r)(\partial_\theta\phi)+(\mathrm{i}/r)(\partial_r\phi).\end{aligned} \tag{47}$$

An interpretation of the Nicolai maps (39) or (47) as stochastic differential equations is only possible by introducing a parameter λ as discussed in § 2. Then the discretisation given by (41) and (42) defines the stochastic differential equation in the sense of Stratonovich.

We conclude this section with a remark on rotationally symmetric systems which will be considered in more detail in the next section. When $\phi(\boldsymbol{q})$ in (18) only depends on $r=|\boldsymbol{q}|$ the Schrödinger equations (17) are separable in polar coordinates. Writing

$$\psi_j(\boldsymbol{q},t)=\sum_{l=-\infty}^{+\infty}\exp[\mathrm{i}(l+1)\theta]r^{-1/2}u_{j,l}(r,t) \tag{48}$$

one has that $u_{1,l}$ and $u_{2,l}$ satisfy

$$\mathrm{i}\dot u_{1,l}=A_l^*A_lu_{1,l}\equiv H_{1,l}u_{1,l} \tag{49}$$

$$\mathrm{i}\dot u_{2,l}=A_{l+1}A_{l+1}^*u_{2,l}\equiv H_{2,l}u_{2,l} \tag{50}$$

with ($\phi'\equiv\partial_r\phi$)

$$A_l=-\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l+\frac12}{r}-\phi' \tag{51}$$

$$A_l^*=\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l+\frac12}{r}-\phi'. \tag{52}$$

The Fokker-Planck equations associated with (49) and (50) via the procedure outlined in § 2 are

$$\dot W_{1,l}(r,\tau)=\frac{\partial}{\partial r}\left(\frac{l-\frac12}{r}+\phi'\right)W_{1,l}(r,\tau)+\frac12\frac{\partial^2}{\partial r^2}W_{1,l}(r,\tau) \tag{53}$$

$$\dot W_{2,l}(r,\tau)=-\frac{\partial}{\partial r}\left(\frac{l+\frac12}{r}+\phi'\right)W_{2,l}(r,\tau)+\frac12\frac{\partial^2}{\partial r^2}W_{2,l}(r,\tau). \tag{54}$$

The associated stochastic differential equations

$$\dot{r}=(-1)^j\left(\frac{l+(-1)^j\frac{1}{2}}{r}+\phi'\right)+\eta \qquad j=1,2 \tag{55}$$

play the role of a Nicolai map in the path integral solution of (49) and (50), respectively. These path integrals can be obtained by making the change of variables to polar coordinates in the discrete representation (27) and (28) with J_k^N given by (40) and integrating out the angular variables.

4. Witten index

In this section we first make some remarks about the existence of a Nicolai map for the Witten index of two-dimensional magnetic field systems. We then discuss its calculation starting from its path integral representation, using scaling properties and topological invariance. We also give a complete scattering theory treatment for a rotationally symmetric system, employing recent results on low-energy scattering [19, 20]. These discussions amplify the results appearing in [10, 20, 21], giving new important details of the problem as we will see in the course of the derivations.

In supersymmetric quantum mechanics, the Witten index [8], Δ, counts the difference betweeen the number of bosonic and fermionic zero-energy modes of the Hamiltonian. If the threshold of the continuous spectrum of $H_1(H_2)$ extends down to zero, which is the case in the systems we consider, the Witten index has to be regularised and one writes, for example, in the heat kernel regularisation

$$\Delta=\lim_{\beta\to\infty}\Delta(\beta) \tag{56}$$

$$\Delta(\beta)=\operatorname{Tr}[\exp(-\beta H_1)-\exp(-\beta H_2)]. \tag{57}$$

Following the treatment in § 3 we arrive at a formal expression for $\Delta(\beta)$, namely

$$\Delta(\beta)=\int_{\mathrm{PBC}}\mathscr{D}\boldsymbol{q}(t)\left[\exp\left(-\tfrac{1}{2}\int_0^\beta \mathrm{d}t\, b(\boldsymbol{q})\right)-\exp\left(\tfrac{1}{2}\int_0^\beta \mathrm{d}t\, b(\boldsymbol{q})\right)\right]$$
$$\times\exp\left(-\int_0^\beta \mathrm{d}t[\tfrac{1}{2}\dot{q}^\mu(t)\dot{q}^\mu(t)-\mathrm{i}\dot{q}^\mu(t)a^\mu(\boldsymbol{q})]\right) \tag{58}$$

where, in comparison with (27), due to the extra integration because of the trace

$$\int_{\mathrm{PBC}}\mathscr{D}\boldsymbol{q}(t)\leftrightarrow\int\prod_{i=0}^{N}\left(\frac{\mathrm{d}\boldsymbol{q}_i}{2\pi\varepsilon}\right). \tag{59}$$

Concerning the existence of a Nicolai map for the Witten index (58), we have found that the transformation (39) and (41) leads to a Jacobian zero in the case of $\Delta(\beta)$, again because of the extra integration coming from the trace. (This is in contrast with the one-dimensional Witten model for which one can show that the Nicolai map for a matrix element of the type (26) is also a Nicolai map for the corresponding Witten index.) In this context we mention that in [22] it has been shown that, by allowing complex gauge transformations, a Nicolai map can be found for (58) in the sense that the fermionic determinant is cancelled by the Jacobian and the bosonic action becomes bilinear. After certain approximations, $\Delta(\beta)$ can be calculated in this way to find

$$\Delta(\beta)=\Delta=-F \tag{60}$$

where F, the flux, is given by

$$F=(2\pi)^{-1}\int \mathrm{d}^2 q b(\boldsymbol{q}). \tag{61}$$

An alternative procedure to obtain (60) starting from (58) is to use topological invariance and scaling properties. Firstly, it has been shown that the regularised Witten index is invariant under sufficiently small perturbations of the operator A (see (1)), i.e. under compact deformations of the potential. This result, called topological invariance, has been demonstrated in the heat kernel regularisation (56) and (57) in [23] and in the resolvent regularisation in [10] (we refer to the latter for exact details on the nature of the perturbations). Secondly, we derive some scaling properties for $\Delta(\beta)$. For this purpose, we concentrate on a specific model which has been considered first in [24] in connection with the study of the nature of the Dirac spectrum in the presence of localised gauge vortices (see also [10, 21] and references therein). In particular, we take

$$\begin{aligned} \boldsymbol{a} &= (\partial_2\phi, -\partial_1\phi) \\ \phi(r,R) &= \begin{cases} -Fr^2/2R^2 & \text{if } r\leqslant R \\ -(F/2)[1+\ln(r^2/R^2)] & \text{if } r\geqslant R \end{cases} \end{aligned} \tag{62}$$

such that

$$b=-\Delta\phi=\begin{cases} 2F/R^2 & r\leqslant R \\ 0 & r>R \end{cases} \tag{63}$$

and (61) is satisfied. Defining the unitary group of dilations in two dimensions to be

$$(U_s g)(\boldsymbol{q})=s^{-1}g(\boldsymbol{q}/s) \qquad s>0 \tag{64}$$

with g a square integrable function, it is straightforward to check that in this case the Hamiltonians H_j, which now depend on R, satisfy

$$U_s H_j(R) U_s^{-1}=s^2 H_j(sR) \qquad j=1,2. \tag{65}$$

This result immediately implies, starting from (58), that

$$\Delta(\beta,R)=\Delta(\beta/s^2, R/s) \tag{66}$$

since we know that $\int_{\mathrm{PBC}}\mathscr{D}\boldsymbol{q}(t)$ is invariant under this scaling (see, e.g., (59)).

Together with the topological invariance, that can be written in the form

$$\Delta(\beta,R)=\Delta(\beta,R/s) \tag{67}$$

(66) implies that Δ cannot depend on R, such that

$$\Delta(\beta)=\Delta(\beta/s^2) \tag{68}$$

or, in other words, Δ is β independent. We then compute the constant value of Δ by using heat kernel expansion techniques for $\beta\to 0$ [10, 25] to find the result (60). It would be interesting to see if this method can be successfully applied to higher-dimensional models. At this point we recall that the two-dimensional magnetic field system is not Fredholm (the operator A is Fredholm if and only if the infimum of the essential spectrum of A^*A is strictly positive), such that the Fredholm index, an integer defined by $i(A)\equiv[\dim\ker(A)-\dim\ker(A^*)]=[\dim\ker(H_1)-\dim\ker(H_2)]$ differs, in general, from the Witten index Δ.

Especially these non-Fredholm systems have been studied recently in the resolvent regularisation scheme ([10, 21] and references therein). In particular the model (62) has been treated by scaling techniques similar to those discussed above. In the rest of this section we want to present a complete scattering theory description of this model, employing Jost function techniques [26] and new results on low-energy scattering [10, 20]. The advantage of this method is that it tells us precisely which zero-energy states contribute to Δ and why some states give non-integer contributions. After finishing this calculation, we became aware of a computation of the anomaly for a system of fermions in two (spatial) dimensions interacting with a localised rotationally symmetric magnetic field, where similar scattering techniques are exploited [27] (see also [28]).

The resolvent regularised Witten index is [29]

$$\Delta = \lim_{z\to 0} \Delta(z) \tag{69}$$

$$\Delta(z) = -z\,\mathrm{Tr}[(H_1 - z)^{-1} - (H_2 - z)^{-1}]. \tag{70}$$

For spherically symmetric systems it is known that these formulae lead to

$$\Delta = \frac{1}{\pi}\sum_l \delta_{12,l}(0) \tag{71}$$

where $\delta_{12,l}(0)$ is the phase shift for the scattering system $(H_{1,l}, H_{2,l})$, normalised to 0 at infinite energy, at energy zero. As we will see the RHS of (71) can be calculated by Levinson's theorem. The two-dimensional magnetic field model (62) can be described by the following set of Schrödinger equations (recall (48)-(52)):

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(l+\frac{1}{2})(l+\frac{3}{2})}{r^2} - \frac{2l_jF}{R^2} + \frac{F^2r^2}{R^4}\right)F_{j,l}(k,r) = k^2F_{j,l}(k,r) \qquad r\leqslant R \tag{72}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(l-F+\frac{1}{2})(l-F+\frac{3}{2})}{r^2}\right)F_{j,l}(k,r) = k^2F_{j,l}(k,r) \qquad r> R \tag{73}$$

with $j=1,2$, $l_1=l$, $l_2=l+2$ and the partial wavenumber l running from $-\infty$ to $+\infty$, and where $F_{j,l}(k,r)$ denote the regular solutions. (The $F_{j,l}(k,r)$ are the time-independent factors of the $u_{j,l}$ in (49) and (50).) We remark that the relative scattering problem $(H_{1,l}, H_{2,l})$ is short-range: $H_{1,l}$ and $H_{2,l}$ only differ by a constant factor $(4F)/R^2$ in the inside region.

We now use standard scattering techniques [26] to solve this problem. Most of the algebra is tedious but straightforward such that we only give a short description of the results.

The solutions of (72) and (73) for $l\geqslant 0$ are given by

$$F_{l,j}(k,r) \overset{r\leqslant R}{=} r^{l+3/2}\exp(-|F|r^2/2R^2)\,{}_1F_1\left(\frac{l}{2}+1-(\mathrm{sgn}\,F)\frac{l_j}{2}-\frac{k^2R^2}{4|F|}, l+2, \frac{|F|r^2}{R^2}\right) \tag{74}$$

$$F_{j,l}(k,r) \overset{r>R}{=} c_j(k,R)r^{1/2}J_{l-F+1}(kr) + d_j(k,R)r^{1/2}Y_{l-F+1}(kr) \tag{75}$$

where ${}_1F_1$ is the confluent hypergeometric function and J_L and Y_L are Bessel functions of order L and where c_j and d_j are determined by continuity of $F_{j,l}$ and its derivative at $r=R$. In the following we denote $L=l-F+1$. Looking at the limit $r\to\infty$ of $F_{j,l}(kr)$

and comparing with the 'free' solution $F_{0,l}(kr)$ that satisfies (73) but now in all space, i.e. for all $r \gtrless R$, the phase shift $\delta_{j,l}(k)$ can be obtained as usual from

$$\tan \delta_{j,l}(k) = -\frac{d_j(k,R)}{c_j(k,R)}. \tag{76}$$

Next we know that the irregular solution of our Schrödinger problem for $l \geqslant 0$ is given by

$$G_{j,l}(k,r) = r^{l+3/2} \exp(-|F|r^2/2R^2)\left[a_{j\,1}F_1\left(\frac{l}{2}+1-(\operatorname{sgn} F)\frac{l_j}{2}-\frac{k^2R^2}{4|F|}, l+2, \frac{|F|r^2}{R^2}\right)\right.$$
$$\left. + b_j U\left(\frac{l}{2}+1-(\operatorname{sgn} F)\frac{l_j}{2}-\frac{k^2R^2}{4|F|}, l+2, \frac{|F|r^2}{R^2}\right)\right] + \alpha_j r^{1/2} H_L^{(2)}(kr) \tag{77}$$

with U the irregular confluent hypergeometric function and $H_L^{(2)}$ the Hankel function, where a_j and b_j are again determined by continuity requirements and α_j by the requirement that the Wronskian $W(G_{0,l}, F_{0,l})_r = 1$. Finally, the Jost function $\mathscr{F}_{j,l}(k)$ for $l \geqslant 0$ is given by

$$\mathscr{F}_{j,l}(k) = W(G_{j,l}, F_{j,l})$$
$$= W\left[r^{l+3/2} \exp(-|F|r^2/2R^2)\,{}_1F_1\left(\frac{l}{2}+1-(\operatorname{sgn} F)\frac{l_j}{2}-\frac{k^2R^2}{4|F|}, l+2, \frac{|F|r^2}{R^2}\right),\right.$$
$$\left.\frac{\mathrm{i}\pi}{2}(\Gamma(L+1))^{-1}\left(\frac{k}{2}\right)^L r^{1/2} H_L^{(2)}(kr)\right]_{r=R}. \tag{78}$$

We now want to study the $k \to 0$ behaviour of the Jost function $\mathscr{F}_{j,l}(k)$. We start with $j=1$. The limit $k \to 0$ of the ${}_1F_1$ function is not transparent. The trick is then to use the known relation

$$F_{1,l}(0,r) = G_{1,l}(0,r) \int_{r_0}^{r} \mathrm{d}t [G_{1,l}(0,t)]^{-2} \tag{79}$$

such that one obtains for the first term in the Wronksian (78) the following formulae:

$$(2l+2)r^{-l-1/2} \exp(Fr^2/2R^2) \int_0^r \mathrm{d}t\, t^{2l+1} \exp(-Ft^2/R^2) + \mathrm{O}(k^2). \tag{80}$$

Using then the $k \to 0$ behaviour of the Hankel function one arrives at the results

$$\mathscr{F}_{1,l}(k) \overset{k\to 0}{=} -[\ln(\tfrac{1}{2}kR)](2l+2)R^{l+1}\exp(-F/2) + \mathrm{O}(k^0) \qquad L=0$$
$$= \frac{2l+2}{2|L|} R^F \exp(-F/2) + \begin{cases} \mathrm{O}(k^{2|L|}) & 0<L<1 \\ \mathrm{O}(k^2) & L \geqslant 1 \end{cases}$$
$$= \frac{2l+2}{2|L|} R^{-|L|+l+1} \exp(-F/2)$$
$$+ (2l+2)R^{-|L|-l-1}\exp(F/2) \int_0^R \mathrm{d}t\, t^{2l+1} \exp(-Ft^2/R^2)$$
$$+ \begin{cases} \mathrm{O}(k^{2|L|}) & 0>L>-1 \\ \mathrm{O}(k^2) & L \leqslant -1. \end{cases} \tag{81}$$

The same study can be made for $j=2$ starting directly from (78) since the ${}_1F_1$ term is now, in the limit $k \to 0$,

$$r^{l+3/2} \exp(-Fr^2/2R^2) + \mathrm{O}(k^2). \tag{82}$$

One obtains

$$
\begin{aligned}
\mathscr{F}_{2,l}(k) &\overset{k\to 0}{=} \exp(-F/2)R^F + \mathrm{O}(k^2 \ln k) \qquad L=0 \\
&= \exp(-F/2)R^F + \begin{cases} \mathrm{O}(k^{2|L|}) & 0<L<1 \\ \mathrm{O}(k^2) & L\geq 1 \end{cases} \\
&= (\tfrac{1}{2}k)^{2|L|} \frac{|L|\pi \exp(|L|\pi \mathrm{i})}{(\sin|L|\pi)(\Gamma(|L|+1))^2} \exp(-F/2)R^{l+1+|L|} + \mathrm{O}(k^2) \qquad 0>L>-1 \\
&= \mathrm{O}(k^2) \qquad L\leq -1.
\end{aligned} \tag{83}
$$

Interpreting these results [19, 20, 30, 31] we see from (81) that $H_{1,l}$ for $l\geq 0$, has a typical generic behaviour at $k=0$, i.e. there are no zero-energy resonances or zero-energy bound states. From (83) we learn that for $H_{2,l}$, $l\geq 0$, we get a two-dimensional s wave resonance behaviour for $L=0$. For $L>0$ we have again generic behaviour while for $L<0$ zero-energy bound states occur if $|L|\geq 1$. However, if $0<|L|<1$ in this case we see a behaviour of the type $k^{2|L|}$. The latter is an explicit realisation of the results discussed in [20].

The same analysis can be made for $l<0$. We do not give any detailed formulae, we just remark that some care is needed when treating $H_{2,l}$ for $l=-1$. For $H_{1,l}$, $l<0$ we get the behaviour of $H_{2,l}$, $l\geq 0$ with L replaced by $-L$. For $H_{2,l}$, $l<0$ we get the behaviour of $H_{1,l}$, $l\geq 0$ so there are no zero-energy resonances or bound states, except for $l=-1$. In the latter case we do get zero-energy bound states for $L=-F<0$ and $|F|>1$ while we see again the $k^{2|L|}$ behaviour for $|F|<1$.

This completely describes the possibilities of having zero-energy states. Since $H_{1,l}$ and $H_{2,l}$ are non-negative there are no negative-energy bound states. In order to extract Levinson's theorem from this information we have to check the normalisation of the phase shift $\delta_{j,l}(k)$ by looking at its behaviour for $k\to\infty$. This can be done, e.g., employing (76), the asymptotic behaviour of ${}_1F_1$ with respect to the first parameter [32] and the asymptotic behaviour of Bessel functions. The result is

$$
\delta_{j,l}(k) \overset{k\to\infty}{=} \begin{cases} (\pi/2)[|L|-l-1]+m_{j,l}\pi & l\geq 0 \\ (\pi/2)[|L|+l+1]+m'_{j,l}\pi & l<0 \end{cases} \qquad m_{j,l},\, m'_{j,l}\in Z,\, j=1,2. \tag{84}
$$

Since the relative $(H_{1,l}, H_{2,l})$ scattering problem is short-range, we can require, without loss of generality, the relative phase shift $\delta_{12,l}(k)$ to be defined such that $m_{1,l}=m_{2,l}$ and $m'_{1,l}=m'_{2,l}$.

Using all the foregoing information Levinson's theorem can then be read off from the following table of contributions. We take, e.g., $F>0$ and we recall $L=l+1-F$.

$\delta_{j,l}(0)$	$l<0$, $l\neq -1$, $L<0$	$l=-1$, $L=-F<0$	$l\geq 0$, $L<0$	$l\geq 0$, $L>0$
$j=1$	0	0	0	0
$j=2$	0	$\begin{cases} \pi F & \text{for } 0<F<1 \\ \pi & \text{for } F\geq 1 \end{cases}$	$\begin{cases} \pi\lvert L\rvert & \text{for } 0<\lvert L\rvert<1 \\ \pi & \text{for } \lvert L\rvert\geq 1 \end{cases}$	0

We calculate

$$
\begin{aligned}
\sum_{l=-\infty}^{+\infty} [(\delta_{1,l}(0)-\delta_{1,l}(\infty))-(\delta_{2,l}(0)-\delta_{2,l}(\infty))] &= \sum_{l=-\infty}^{+\infty} (\delta_{1,l}(0)-\delta_{2,l}(0)) \\
&= -\pi[1+([F]-1)+(F-[F])] \\
&= -\pi F
\end{aligned} \tag{85}
$$

where we have used (84) in the first step and $[F]$ denotes the integer part of F. So by using (71) we find (60). We thereby explicitly see that only the partial waves starting from $l=-1$ up to $l<F-1$ contribute to the Witten index. Those with $l=-1$ up to $l=[F]-2$ are zero-energy bound states and they each contribute 1; the highest allowed partial wave, $l=[F]-1$, contributes $(F-[F])$. The latter corresponds to the $k^{2|L|}$ behaviour of $\mathscr{F}_{2,l}(k)$ at $k\to 0$ found in (83).

Acknowledgments

We gratefully acknowledge stimulating discussions with F Gesztesy, R Graham, H Grosse and A Van Proeyen.

Appendix

The Lagrangian (33)-(35) can be obtained in the superspace formalism by considering constrained supervariables as follows (see also [33]).

Let ϕ^μ, $\mu=1,2$ be real supervariables defined on the superspace parametrised by (t, θ, θ^*), namely

$$\phi^\mu(t,\theta,\theta^*)=q^\mu(t)+\theta\psi^\mu(t)+\psi^{*\mu}(t)\theta^*+\theta^*\theta f^\mu(t) \qquad \mu=1,2. \tag{A1}$$

Instead of treating $\boldsymbol{q}$, ψ, ψ^* and f as independent variables as was done in [14] we choose to impose the following constraint, compatible with supersymmetry:

$$D_\theta(\phi^1+\mathrm{i}\phi^2)\equiv 0 \tag{A2}$$

where D_θ is the invariant derivative

$$D_\theta=\frac{\partial}{\partial\theta}-\mathrm{i}\theta^*\frac{\partial}{\partial t}. \tag{A3}$$

In the language of field theory this makes $(\phi^1+\mathrm{i}\phi^2)$ a 'chiral' supervariable.

Equation (A2) is equivalent to the following relations between the components of ϕ^1 and ϕ^2:

$$\begin{aligned}&\dot q^1+f^2\equiv 0\\ &\dot q^2-f^1\equiv 0\\ &\psi^1+\mathrm{i}\psi^2\equiv 0\\ &\psi^{*1}-\mathrm{i}\psi^{*2}\equiv 0.\end{aligned} \tag{A4}$$

Imposing these constraints on the supersymmetric Lagrangian

$$\begin{aligned}L=&\tfrac{1}{2}\dot q^\mu\dot q^\mu+\tfrac{1}{2}f^\mu f^\mu+f^\mu W_{,\mu}+\tfrac{1}{2}\mathrm{i}(\dot\psi^\mu\psi^{*\mu}-\psi^\mu\dot\psi^{*\mu})\\ &+\tfrac{1}{2}W_{,\mu\nu}(\psi^{*\mu}\psi^\nu-\psi^\mu\psi^{*\nu})\end{aligned} \tag{A5}$$

(where the notation $_{,\mu}$ means the derivative with respect to q^μ) which have been derived in [14] from the superspace expression

$$\mathscr{L}(\phi, D_\theta\phi)=\tfrac{1}{2}(D_\theta\phi^\mu)(D_\theta\phi^\mu)^*+W(\phi) \tag{A6}$$

and introducing the notation $\psi=\psi^1$, $\phi=\frac{1}{2}W$ and the Euclidean time $\tau=\mathrm{i}t$ one obtains the Lagrangian of the magnetic field systems.

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